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On Chern-Simons Quivers and Toric Geometry

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Abstract

We discuss a class of 3-dimensional $\mathcal{N} = 4$ Chern-Simons (CS) quiver gauge models obtained from M-theory compactifications on singular complex 4-dimensional hyper-Kähler (HK) manifolds, which are realized explicitly as a cotangent bundle over two-Fano toric varieties V^2. The corresponding CS gauge models are encoded in quivers similar to toric diagrams of V^2. Using toric geometry, it is shown that the constraints on CS levels can be related to toric equations determining V^2.

Keywords: 2D $\mathcal{N} = 4$ sigma models, M-theory, Chern-Simons quivers, toric geometry.

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1 Introduction

In string theory, the properties of gauge quiver sectors localized on worldvolume of D-branes at singular points of Calabi-Yau manifolds in the low energy theory are determined by the local geometry around those singularities [1, 2]. In the $\mathcal{N} = 1$ supersymmetric case for instance, the field content of D-brane worldvolumes can be encoded in a quiver diagram, where the i^{th} node represents the U(N_i) factor of the gauge group and there are oriented arrows from the i^{th} to the j^{th} node corresponding to $\mathcal{N} = 1$ chiral multiplets in the $(N_i, \overline{N}_j)$ representation [3]. The matter content is given by bi-fundamental hypermultiplets in the representations $(N_i, \overline{N}_j)$ of the gauge group. The superpotential is obtained by restricting the chiral multiplets to the invariant ones. The moduli spaces of the vacua associated with such quivers that corresponds to the space of possible locations of the Dp-brane, are usually isomorphic to orbifold geometries. In connection with AdS/CFT correspondence in four dimensions, the dual models are given by spaces of the form $AdS_5 \times M_5$, where $M_5 = S^5/\Gamma$ with Γ being a discrete subgroup of SO(6) [4]. In particular, these can be obtained by placing D3-branes at ADE singularities of Calabi-Yau threefolds generalizing the conifold singularity [5].

Recently, 3-dimensional Chern-Simons (CS) quivers have attracted much attention and have been investigated from various points of view in type II superstrings and M-theory compactifications [6, 7, 8, 9]. In particular, it has been pointed out that 3-dimensional $\mathcal{N} = 6$ CS quiver with U$(N)_k \times$ U$(N)_{-k}$ gauge symmetry is dual to M-theory propagating on $AdS_4 \times S^7/Z_k$, with an appropriate amount of fluxes, or to type IIA superstring on $AdS_4 \times \mathbf{CP^3}$ for large number of k, N with $k \geq N$ corresponding to the weakly interacting regime [10]. In the decoupling limit, the corresponding CFT$_3$ is generated by the action of multiple M2-branes placed at the orbifold C^4/Z_k. This result can be considered a nice example for understanding the AdS/CFT correspondence in three dimensions.

This analysis has been extended in several ways and applied for a large class of examples of CS quivers which are proposed to be dual theories to theories on non trivial seven dimensional manifolds. In particular, it has been suggested that some $\mathcal{N} = 2$ models with M2's are dual to seven dimensional toric Sasaki-Einstein manifolds Y$_7$, which are considered as the base of a Calabi-Yau cone C(Y$_7$). The corresponding Chern-Simons-Matter (CSM) theories are quivers with different number of supersymmetries and their classical Vacuum Moduli Spaces (VMS) have been analyzed in several works, take for example the case of [6]. In particular, it has also been shown how to extract toric data of the Calabi-Yau 4-fold C(Y$_7$) to describe the corresponding quiver theories. These $\mathcal{N} = 2$ Chern-Simons share many similarities with four-dimensional quiver gauge models preserving only four supercharges, obtained from type II superstrings on Calabi-Yau threefolds or M-theory on G2 manifolds, although their gauge symmetries and D-term conditions are modified [11, 1]. In [12], a realization of Maldacena

conjecture [13] in three dimensions (AdS_4/CFT_3) was searched for. There, it was there pointed out the necessity of having $\mathcal{N} \geq 4$ superconformal CS theories to capture the low energy description of the infrared fixed point of the worldvolume gauge theory describing multiple M2-branes. The $\mathcal{N} = 8$ realization, corresponds to the description of multiple M2-branes in flat space [14]. In the strongly coupled regime the associated physics corresponds to the worldvolume description of multiple M2-branes[1] in distinction with type IIB superstring case that describes the weakly coupled regime. When the 3d theory is dual to M-theory on $AdS_4 \times Y_7$ then the Vacuum Moduli Space (VMS) of the 3d theory $\mathcal{M}_{3d}$ should coincide with the CY 4-fold cone $C(Y_7)$, consisting of a quiver diagram associated with the D-terms, a superpotential and CS levels (k_i).

Gaiotto-Witten (GW) [9] and [7] exhausted all the 3d CSM quiver compatible $\mathcal{N} = 4$ conformal supersymmetries. For other formulations developped as twisted examples of these two types of constructions see [16, 17]. In the case of $\mathcal{N} = 4$ SCM theories the Vacuum Moduli space (VMS) also receives quantum corrections [16, 18] that we do not consider here.

The aim of this work is to contribute to these activities by considering a class of 3-dimensional CSM quiver gauge theories from geometric data of a particular M-theory compactification. In this way, the gauge group and matter content of the resulting models are obtained from the singularities of complex 4-dimensional hyper-Kähler (HK) manifold. The geometry is realized explicitly as a cotangent bundle over complex two-dimensional toric varieties. It enables one to represent the corresponding CS gauge models by quiver diagrams similar to toric graphs of two-Fano toric varieties V^2. One considers the quiver to be composed of a set of vertices, each of which is associated with U(N) gauge group factor, and for each pair of vertices with matter in bi-fundamental representations, where the vertices are connected by lines required by toric geometry. This diagram encodes some information on the world-volume of M2-branes probing the above toric Calabi-Yau four-fold (hyper) cones. As an illustration, we consider triangle and rectangular quiver gauge models by introducing projectives spaces, Hirzebrouch surfaces and del Pezzo surfaces in the base of the cotangent fibration in M-theory backgrounds. Using toric geometry, we show that the CS constraints can be converted into toric equations defining two-Fano toric varieties. We have taken as a departing point, the D-term constrains of the conformal SCM (Super Chern-Simons Matter) theory to construct a quiver and together with the superpotential (F-term constrains), to obtain a toric Hyperkahler $\mathcal{N} = 4$ diagram which has only two directions compact. We compare these constructions with the GW models, by imposing restrictions on the allowed values for the CSM levels.

[1]For a High Energy description (opposite regime to the decoupling limit) of multiple M2-branes with an arbitrary number of colors and without imposing confomality see [15].

2 Quiver Chern-Simons Theories

In [19, 20], a $\mathcal{N} = 8$ M2-branes lagrangian was constructing trying to describe the IR limit of a 3-dimensional CFT (CFT_3) to realize the Maldacena conjecture in three dimensions ($\mathrm{AdS}_4/\mathrm{CFT}_3$) but restricted just to $\mathcal{N} = 2$ branes. However, the difficulty of this correspondence is due the lack of understanding of a IR fixed point of a conformal field theory of multiple of M2-branes, in particular probing Calabi-Yau fourfolds. It was pointed out that it was needed a supersymmetry $\mathcal{N} \geq 2$ to realize such a limit. The authors of [10], constructed a $\mathcal{N} = 6$ Chern-Simons with $\mathrm{U}(N) \times \mathrm{U}(N)$ as the worldvolume of N M2-branes placed at the C^4/Z_k orbifold. As a quiver diagram, this model, which is known by the ABJM theory, consists of two nodes. To each node we associate a gauge factor $\mathrm{U}(N)$. These have a Chern-Simons lagrangian with levels k and $-k$. The $\mathcal{N} = 4$ Superconformal Chern Simons theories was constructed by [9] with $SO(4)$ R-symmetry and $OSp(4|4)$ conformal symmetry. The associated target space corresponds to a noncompact toric hyperkahler manifold. It corresponds to the description of multiple M2-branes once that the Fundamental Identity[2] is imposed to guarantee invariance under the superconformal Lie algebra. This Fundamental Identity imposes constrains to the Chern Simons level of the gauge groups to appear in alternating pairs $(k, -k)$. In such a way that the unique nonabelian gauge groups for a single type of hypermultiplets, were proved to be: $\mathrm{U}(N)_k \times \mathrm{U}(M)_{-k}$ and $\mathrm{O}(N)_k \times \mathrm{Sp}(M)_{-k}$ or direct sums in blocks of them. Matter $(q_\alpha, \Psi^A_{\dot\alpha})$ are in hypermultiplets in the bifundamental. The associated quiver is linear in such a way that for gauge groups $\mathrm{U}(N_1)_{k_1} \times \mathrm{U}(N_2)_{k_2} \times \mathrm{U}(N_3)_{k_3}$ the levels are imposed to be $k_1 = k, k_2 = 0, k_3 = -k$. The associated toric manifolds has been shown to be:

$$\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \mathrm{U}(N_3)/\mathrm{U}(N_2) \tag{2.1}$$

In general for an arbitrary number of gauge groups the associated quiver diagram has $k_1 = k, k_2 = k_3 = \cdots = k_{s-1} = 0, k_s = -k$. In [7], the GW theory was generalized by adding a twisted hypermultiplet to the GW construction in order to include links among the different pairs of nodes. The two types of hypermultiplets alternate among the gauge groups. The resulting quiver can be linear or circular with multiple nodes. The simplest case corresponds to the BLG model with a modified R-symmetry $SO(4)$. When the gauge groups are $\mathrm{U}(N_i)$, the quivers are considered as the Dynkin diagrams of A_n series and $K^{mn} = (-1)^I k$ with $I \in Z$. The authors of [9] and [7] classify all the toric manifolds for a GLSM with $\mathcal{N} = 4$ representing the low energy description of multiple M2 branes on the hyperkahler construction.

In [16], it is analyzed a case of a $\mathcal{N} = 4$ CS theory with auxiliary vector multiplets. It has been considered quiral fields in the bifundamental and the Chern Simons coupling generalized to be $k_I = \frac{k}{2}(s_I - s_{I-1})$ with $s_I = \pm 1$ $k > 0$ allowing then some of the levels to vanish, $k_I = 0$.

[2]The FI condition is the generalization of Jacobi indentity to Fillipov algebras.

This breaks the superconformal Lie algebra and the dual theory is no longer associated with M2-branes but with type IIB superstring in the presence of N D3-branes intersecting on a circle S^1 and n 5-branes intersecting the D3 along the S^1. The nonlinear sigma models can be constructed as GW model indicated by performing the hyperkahler quotient of all nodes of the corresponding linear quiver diagram except for those gauge groups of the extreme. In [17], it has been given a recipe to construct the general nonlinear sigma models with compact gauge group and whose target space is the contangent fibration of a flag manifold $T^*(CP^1)$ for hyperkahler $\mathcal{N} = 4$ superconformal CSM theories in $d = 3$.

There have appeared many models extending ABJM, and describing three dimensional $\mathcal{N} = 2$ quiver Chern-Simon theories. They are conjectured to be gauge field duals of AdS_4 background in type IIA superstring and M-theory compactifications. These CS quiver theories can be interpreted in terms of M2-branes placed at toric four-folds singularities. The simple model is described by an abelian gauge symmetry $\mathrm{U}(1)_{k_1} \times \mathrm{U}(1)_{k_2} \times \ldots \times \mathrm{U}(1)_{k_n}$, where k_i denote Chern-Simons levels for each factor U(1). These models can be encoded in a graph formed by n vertices where each gauge group factor U(1) is associated with a vertex while the matter is represented by the link between vertices. Using similarity with $\mathcal{N} = 1$ quivers in four dimensions, the corresponding toric moduli space have been discussed in [6]. Among others, it has been shown that Chern-Simons levels k_i are constrained by

$$\sum_i k_i = 0. \tag{2.2}$$

This is a necessary condition for the moduli space to be a four complex dimensional. For the general case where the gauge group is $\prod_i \mathrm{U}(N_i)_{k_i}$, the constraints on k_i read as

$$\sum_i k_i N_i = 0. \tag{2.3}$$

In what follows, we will show that these constraints can be solved using toric geometry equations.

3 Toric description of Chern-Simons quivers

In this section we give a toric description of a class of CS quivers obtained from M-theory on hyperkahler backgrounds. In particular, we will show that the CS conditions given in eq.(2.2) and eq.(2.3) can be translated into nice toric algebraic equations [21, 22]. The latters appear in the standard construction of complex manifolds using toric geometry. Recall that, a n-dimensional toric variety $\mathbf{V^n}$ can be represented by a toric diagram (polytope) $\Delta(\mathbf{V}^n)$ spanned

by $k = n + r$ vertices v_i of an $\mathbf{Z^n}$ lattice satisfying

$$\sum_{i=1}^{n+r} q_i^a v_i = 0, \quad a = 1, \ldots, r, \tag{3.1}$$

where q_i^a are integers. For each a they form the so-called Mori vectors. The simplest example in toric geometry, which turns out to play a crucial role in the building blocks of higher-dimensional toric varieties, is $\mathbf{CP^1}$. This geometry has an U(1) toric action ($z \to e^{i\theta} z$) with two fixed points v_1 and v_2 on the real line. These two points satisfy the following condition

$$v_1 + v_2 = 0, \tag{3.2}$$

and they describe respectively the north and south poles of $\mathbf{CP^1}$. The corresponding polytope is just the segment $[v_1, v_2]$ joining the two points v_1 and v_2. Thus, $\mathbf{CP^1}$ can be viewed as a segment $[v_1, v_2]$ with a circle on top, where the circle vanishes at the end points v_1 and v_2. For higher dimensional geometries, the toric descriptions are slightly more complicated. For more details see [21, 22].

Roughly speaking, we will be interested in CS quivers associated with gauge theories obtained from M-theory on the cotangent bundle over two dimensional toric variety $\mathbf{V^2}$. A nice way to describe such M-theory backgrounds is to use the so-called hyper-Kähler quotient studied in [23, 24] engineered by considering a two-dimensional $\mathrm{U}(1)^r$ sigma model with eight supercharges ($\mathcal{N} = 4$) and $r + 2$ hypermultiplets. There is a $\mathrm{SU}(r + 2)$ global symmetry under which the hypermultiplets transform in the fundamental representation $r + 2$. This background is defined by the following D-flatness condition

$$\sum_{i=1}^{r+2} q_i^a [\phi_i^\alpha \bar{\phi}_{i\beta} + \phi_i^\beta \bar{\phi}_{i\alpha}] = \vec{\xi}_a \vec{\sigma}_\beta^\alpha, \quad a = 1, \ldots, r \tag{3.3}$$

where q_i^a is a matrix charge specified later on. ϕ_i^α's ($\alpha = 1, 2$) denote the component field doublets of each hypermultiplets ($i = 1 \ldots, r + 2$). $\vec{\xi}_a$ are the Fayet-Illiopoulos (FI) 3-vector couplings rotated by the SU(2) symmetry, and $\vec{\sigma}_\beta^\alpha$ are the traceless 2×2 Pauli matrices. Performing SU(2) R-symmetry transformations $\phi^\alpha = \varepsilon^{\alpha\beta}\phi_\beta$, $\overline{\phi^\alpha} = \overline{\phi}_\alpha$, $\varepsilon_{12} = \varepsilon^{21} = 1$ and replacing the Pauli matrices by their expressions, the identities (3.3) can be split as follows

$$\begin{aligned}
\sum_{i=1}^{r+2} q_i^a (|\phi_i^1|^2 - |\phi_i^2|^2) &= \xi_a^3 \\
\sum_{i=1}^{r+2} q_i^a \phi_i^1 \overline{\phi}_i^2 &= \xi_a^1 + i{\xi^2}_a \\
\sum_{i=1}^{r+2} q_i^a \phi_i^2 \overline{\phi}_i^1 &= \xi_a^1 - i{\xi^2}_a.
\end{aligned} \tag{3.4}$$

Using the fact that the resulting space of (3.4) is invariant under $\mathrm{U}(1)^r$ gauge transformations, we get precisely an eight-dimensional toric HK manifold. However, explicit solutions of these

geometries depend on the values of the FI couplings. Taking $\xi_a^1 = \xi_a^2 = 0$ and $\xi_a^3 > 0$, (3.4) describe the cotangent bundle over complex two-dimensional toric varieties. Indeed, if we set all $\phi_i^2 = 0$, we get two-dimensional nonsingular toric variety $\mathbf{V^2}$ defined by

$$\sum_{i=1}^{2+r} q_i^a |\phi_i^1|^2 = \xi_a^3, \quad a = 1, \dots, r. \tag{3.5}$$

To connect this equation with toric geometry, we associate to each field ϕ_i^1 a vector $v_i = (v_i^1, v_i^2)$ in the standard lattice $\mathbf{Z^2}$, such that the v_i fulfill the following relations

$$\sum_{i=1}^{2+r} q_i^a v_i = 0, \quad a = 1, \dots, r. \tag{3.6}$$

Up to eq.(3.6), these $k = 2 + r$ vertices v_i represent the toric diagram Δ of $\mathbf{V}^2$ ($\Delta(\mathbf{V}^2)$). The last equations of (3.4) mean that the ϕ_i^2's define the cotangent fiber directions over $\mathbf{V^2}$. To see that, let us illustrate the idea by using the leading example: $T^*(\mathbf{CP^2})$ contangent bundle. Indeed, we consider $2d$ $N = 4$ supersymmetric U(1) gauge theory with one isotriplet FI coupling $\vec{\xi} = (\xi^1, \xi^2, \xi^3)$ and only three hypermultiplets of charges $q_a^i = q^i = 1; i = 1, 2, 3$. The zero energy states of this gauge model are obtained by solving

$$\begin{aligned}
\sum_{i=1}^{3} (|\phi_i^1|^2 - |\phi_i^2|^2) \quad &= \xi^3 \\
\sum_{i=1}^{3} \phi_i^1 \overline{\phi}_{i2} \quad &= \xi^1 + i\xi^2 \\
\sum_{i=1}^{3} \phi_i^2 \overline{\phi}_{i1} \quad &= \xi^1 - i\xi^2.
\end{aligned} \tag{3.7}$$

As we have seen before, the solutions of eqs.(3.7) depend on the values of the FI couplings. For the case where $\xi^1 = \xi^2 = \xi^3 = 0$, the moduli space has an $SU(3) \times SU(2)_R$ symmetry; it is a cone over a seven manifold described by

$$\sum_{i=1}^{3} (\varphi_{\alpha i} \overline{\varphi}_i^\beta - \varphi_i^\beta \overline{\varphi}_{\alpha i}) = \delta_\alpha^\beta. \tag{3.8}$$

For the case $\vec{\xi} \neq \vec{0}$, the abovementioned $SU(3) \times SU(2)_R$ symmetry is explicitly broken down to $SU(3) \times U(1)_R$. In the remarkable case where $\xi^1 = \xi^2 = 0$ and ξ^3 positive definite, it is not difficult to see that (3.7) describe the cotangent bundle of $\mathbf{CP^2}$. Putting now $\phi_i^2 = 0$, one gets

$$|\phi_1^1|^2 + |\phi_2^1|^2 + |\phi_3^1|^2 = \xi^3 \tag{3.9}$$

defining now the $\mathbf{CP^2}$ projective space. On the other hand, with $\xi^1 = \xi^2 = 0$ conditions, the two last equations of (3.7) may be interpreted to mean that $\overline{\phi}_{2i}$ lies in the cotangent space to $\mathbf{CP^2}$. This can be viewed as an extension of the canonical complex cone over $\mathbf{CP^2}$ used in the study of $\mathcal{N} = 1$ quivers embedded in type II superstrings [1].

Having constructed eight dimensional toric HK manifolds, the following will be concerned with M-theory on such manifolds. We will try to give a toric geometry description of the corresponding CS quivers. In particular, one can interpret the constraint equations eq.(2.2) and eq.(2.3) as toric geometry equations describing two dimensional toric manifolds V^2. The analysis we will be using here is based on the similarity between the four dimensional quivers and CS ones. To indicate the ideas, we will focus on CS quiver theories associated with sigma models on the cotangent bundle over V^2. Our toric geometry interpretation of this class of models relies on the fact that the corresponding quivers are identified with the toric diagram of V^2. In this way, the gauge group and Chern-Simons levels should be related to toric data (q_i^a, v_i) of V^2. Indeed, a close examination of the quiver method, used in string theory, reveals that there are many similarity between such CS quivers and four dimensional $\mathcal{N} = 1$ quivers embedded in type II superstrings on the line bundle of V^2 using the so called (p, q) brane webs [1, 2]. Motivated by [6] and based on these observations, we give a toric interpretation of M-theory CS quivers. In particular, we will interpret the toric data defining V^2 as constraints on CS levels. In fact precisely, the parameters k_i and N_i given in eq.(2.2) and eq.(2.3) will be identified with the toric data (q_i^a, v_i). To see that, let us discuss a simple example where V^2 is $\mathbf{CP^2}$ defined by eq.(3.6) for $r = 1$ and the vector charge $q_i = (1, 1, 1)$. We will see that other examples may be dealt with in a similar way. The corresponding toric quiver (diagram) has 3 vertices, where each vertex is associated with an $\mathrm{U}(N)$ gauge symmetry factor. For this CS quiver, we have a level vector (k_1, k_2, k_3) such that

$$k_1 + k_2 + k_3 = 0. \tag{3.10}$$

We will show that this constraint can be solved by exploring the $\mathbf{CP^2}$ toric data. Indeed, we start by recalling that $\mathbf{CP^2}$ is a complex two dimensional manifold with an $\mathrm{U}(1)^2$ toric action exhibiting three fixed points v_1, v_2 and v_3. Its polytope Δ_2 is a finite sublattice of the Z^2 square lattice. This polytope is described by the intersection of three $\mathbf{CP^1}$ curves defining a triangle $(v_1 v_2 v_3)$ in the $\mathbf{R^2}$ plane. Toric geometry requires that a convenient choice of the data of the three vertices is constrained by

$$v_1 + v_2 + v_3 = 0. \tag{3.11}$$

This equation looks like the constraint on CS levels given in eq.(3.10). In fact, eq.(3.10) can be solved by

$$k_i = (v_i^1 + v_i^2)k \tag{3.12}$$

where k is an arbitrary integer. It is worth to give some remarks regarding this solution. First, the same result should be also obtained for the dual polytope ${\Delta_2}^\star$. For example, the dual polytope of Δ_2 with vertices $(-1, -1)$, $(-1, 2)$ and $(2, -1)$ corresponding to $\mathbf{CP^2}$ with a degree

3 bundle on it, is $\Delta_2{}^\star$ being bounded by $(1,1)$, $(-1,0)$ and $(0,-1)$. Since the dual polytope is useful for constructing mirror toric varieties, it should be interesting to make contact with the mirror maps in CS theories discussed in [25]. Second, it is straightforward to recover the result concerning $Y^{p,k}$ metrics studied in [6]. For $k = 0$, the connection can be ensured by the solution

$$k_i = p({v^\star}_i^1 + {v^\star}_i^2) \tag{3.13}$$

where now ${v^\star}_i$ are the toric vertices of the dual polytope of $\mathbf{CP^2}$. It worth to note that, in the case in which the Chern Simons levels are associated with a linear quiver with $k_1 = k, k_2 = 0, k_3 = -k$ would correspond to the GW model [9].

As we have seen, the vector (k_1, k_2, k_3) in triangle CS quiver is related to the toric vertices (v_1, v_2, v_3). It is natural to ask many questions. One of them is as follows. Is there any interpretation for the ranks N_i?. In what follows, we speculate on it. For this reason, it may be useful to introduce toric geometries with $q_i \neq 1$. In fact, we will replace $\mathbf{CP^2}$ by a weighted projective spaces $\mathbf{WCP^2}_{q_1,q_2,q_3}$. In this case, eq.(3.11) becomes

$$q_1 v_1 + q_2 v_2 + q_3 v_3 = 0. \tag{3.14}$$

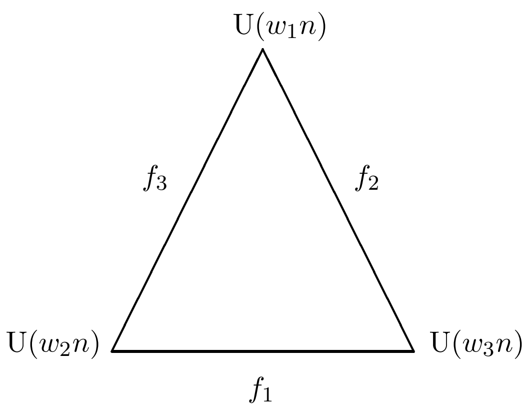
Keeping the same analysis of $\mathbf{CP^2}$, eq.(2.3) can be solved by

$$\begin{aligned} k_i &= (v_i^1 + v_i^2)k \\ N_i &= q_i n \end{aligned} \tag{3.15}$$

where n is arbitrary. The corresponding gauge symmetry is

$$\mathrm{U}(q_1 n) \times \mathrm{U}(q_2 n) \times \mathrm{U}(q_3 n) \tag{3.16}$$

This model is encoded in a quiver with 3 vertices. Each vertex is associated with $\mathrm{U}(q_i n)$ gauge group and the links are associated with Πf_i bifundamental fields. This can be represented by



The numbers f_i are given by

$$f_i = \epsilon_{ijk} q^j q^k d \tag{3.17}$$

where $d=\sum_i q_i$ which is exactly the Calabi-Yau condition in $\mathbf{WCP^2}_{q_1,q_2,q_3}$.

In the end of this section, we note that it is possible to recover the result of GW from the above toric description. As we have mentioned in the introduction, the GW quiver has 3 vertices. Its vector of CS levels is $(k,0,-k)$. Toric geometrically, this model can be obtained by considering V^2 as the canonical line bundle over $\mathbf{CP^1}$ which is given by

$$O(-2)\to \mathbf{CP^1}. \tag{3.18}$$

The toric diagram for this geometry is defined by three vertices (v_0,v_1,v_2) such that

$$-2v_0+v_1+v_2=0. \tag{3.19}$$

which is an open polytope. This can be represented by 3 points in Z^2 given by $v_i=\{(-1,1),(0,1),(1,1)\}$ where the last entry is 1. If we delete this entry we get the mirror geometery described by ${v^\star}_i=\{(-1),(0),(1)\}$ along the line. This toric data can recover the GW quiver with three nodes. In this case, the solution is given by

$$k_i={v^\star}_i k. \tag{3.20}$$

More generally, this model may be extended in a natural way to an 2d $\mathcal{N}=4$ $\mathrm{U}(1)^r$ gauge theory where the moduli space is given by the intersection of r canonical line bundle over $\mathbf{CP^1}$'s. The corresponding quiver may be identified with a bouquet of GW model.

4 More on toric CS quivers

The above analysis may be extended to the case of quivers with more than three vertices. This can be realized by replacing two dimensional projective spaces either by Hirzebrouch surfaces or del Pezzo surfaces whose toric diagrams contain more than three vertices. The corresponding CS quivers involve more than three $\mathrm{U}(N_i)$ gauge symmetry factors. Roughly, Hirzebrouch surfaces $\mathbf{F}_\ell$, for instance, are complex two-dimensional toric surfaces defined by non-trivial fibrations of a $\mathbf{CP^1}$ over a $\mathbf{CP^1}$. This fibration is classified by an integer ℓ, being the first Chern class integrated over $\mathbf{CP^1}$. In $\mathcal{N}=2$ sigma model language, $\mathbf{F}_\ell$ are realized as vacuum manifolds described by $\mathrm{U}(1)\times\mathrm{U}(1)$ gauge theory with four chiral fields with charges

$$q_i^1=(1,1,0,\ell),\qquad q_i^2=(0,0,1,1). \tag{4.1}$$

Toric geometrically, these surfaces are represented by four vertices v_i belonging to $\mathbf{Z^2}$ and satisfying the following toric constraints

$$\begin{aligned} v_1+v_2+\ell v_4&=0\\ v_3+v_4&=0. \end{aligned} \tag{4.2}$$

For simplicity reason, we will restrict ourselves to $\mathbf{F}_0$ describing a trivial fibration of $\mathbf{CP^1}$ over $\mathbf{CP^1}$. In this case, eq.(4.2) reduces to

$$\begin{aligned} v_1 + v_2 &= 0 \\ v_3 + v_4 &= 0. \end{aligned} \tag{4.3}$$

Since we have more than one toric equation, the link discussed in the previous section is not obvious. However, the corresponding quiver can be interpreted as two orthogonal GW quiver models. In terms of CS quivers, eq.(4.3) may be thought of as a particular situation of

$$k_1 + k_2 + k_3 + k_4 = 0 \tag{4.4}$$

describing a rectangular CS quiver with the same gauge group ranks [6]. A priori, this equation can have may solutions. However, toric geometry of two dimensional varieties requires only two possible solutions given by

$$k_i = 0, \qquad \sum_{j\neq i} k_j = 0, \tag{4.5}$$

$$k_i + k_j = 0, \qquad \sum_{k\neq i,j} k_k = 0 \tag{4.6}$$

where $i, j, k, l = 1, 2, 3, 4$. Otherwise, eq.(4.4) can be interpreted as a toric equation of three dimensional projective space $\mathbf{CP^3}$ which is not needed here. It follows from the above discussion that the first solution given in eq.(4.5) corresponds to the CS quiver based on $\mathbf{CP^2}$ toric graph that corresponds to a GW model for two gauge factors with $(k, -k)$, whereas the second one eq.(4.6) should be associated with CS quiver based on F_0 geometry. The latter can be viewed as two orthogonal GW models. Indeed, ignoring the node associated with the zero CS level, the corresponding CS quiver can be identified with two blocks of two gauge factors with $(k, -k)$. The connection with F_0 requires a combination of the Mori vector charges $q_i^a, (a = 1, 2)$. The proposed solution can be given by

$$\begin{aligned} k_i &= (v_i^1 + v_i^2)k \\ N_i &= (q_i^1 + q_i^2)n. \end{aligned} \tag{4.7}$$

It is possible to get the CS quiver based on F_0 by taking the following toric data

$$\begin{aligned} v_1 &= (1, 0) \\ v_2 &= (-1, 0) \\ v_3 &= (0, 1) \\ v_4 &= (0, -1). \end{aligned} \tag{4.8}$$

This toric data produces the quiver CS theory proposed in [6]. The gauge symmetry is given by

$$\mathrm{U}(N) \times \mathrm{U}(N) \times \mathrm{U}(N) \times \mathrm{U}(N) \tag{4.9}$$

where the CS levels are $(k, -k, k, -k)$. This model corresponds to a circular quiver with four nodes, and it would be nice to see if this quiver could be associated with the quiver given in [18].

5 Discussions

In this work, we have discussed 3-dimensional $\mathcal{N} = 4$ CS quiver models. In particular, we have considered a class of such models obtained by the compactification of M-theory background viewed as target space of $\mathcal{N} = 4$ sigma model. The geometry is realized explicitly as cotangent bundles over complex two-dimensional toric varieties. The corresponding quivers are identified with toric graphs of such two-Fano varieties. In particular, we have analyzed in some detail the cases of two dimensional projectives spaces and Hirzebrouch surfaces. Using toric geometry, we have discussed how the physical constraints of this M-theory on backgrounds can be related to toric description of bi-dimensional toric manifolds. We have discussed the possibility to relate them with M2-brane quivers models of the GW construction via restricting the CS constrains to those values that satisfy the Fundamental Identity as indicated in [9].

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